Strategy and Fundraising in Sequential Majoritarian Elections

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July 14, 2020

Motivation

Many strategic interactions are repeated and majoritarian.

- Elections
- Sports series
- R&D Races

Important applications, but analytically non-trivial to find the equilibria.

Structure

Two players compete in campaign of best-of-$2m-1$ 'battles'.
In each battle they allocate some resource ('effort').

The battles might be:

- Sequential or simultaneous
- Deterministic or stochastic
- Unlimited (with MC) or limited resource

Contribution

Simultaneous version well studied (Gross and Wagner (1950), Blackett (1954),Roberson (2006))

Sequential version has been partially analysed:

- Unlimited resource with marginal cost and stochastic battles (Klumpp and Polborn (2006)).
- Unlimited resource with marginal cost and deterministic battles (Konrad (2012)).

But no complete analysis of sequential game with stochastic battles and limited resources!

Model

Two players start with budgets $b_1, b_2 > 0$.

First to m battles. Played in sequence. Define point (i, j) and $n = i + j - 1$.

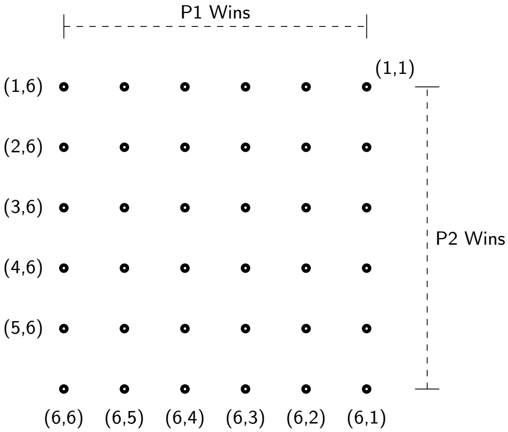
In each battle, if player one chooses x effort, and player 2 chooses y, then player one wins the battle with probability

$$p_1(x, y) = \frac{f(x)}{f(x) + g(y)}.$$

Minimal assumptions on f, g (unlike Fu et al. (2015), Klumpp and Polborn (2006), Konrad (2016).)

▸ Microfoundations

Visualisation



At battle (i, j) players have access to the following information: $(i, j, n, b_1, b_2, f, g)$ and historical expenditures / results.

Symmetric Equilibrium - Intuition

Suppose we are at the point where player 1 needs to win just one more game, and player 2 needs four more.

Suppose each player has a budget of 1.
Technologies: $f(x) = g(x) = x$.

Clear that player 2 should best respond to symmetric strategy symmetrically.

Suppose player 2 is playing symmetrically.

- If player 1 chances it all on today's match. $\mathrm{P(WIN)} = 0.80$
- If player 1 abandons today's battle: $\mathrm{P(WIN)} = 0.92$
- If player 1 players symmetrically: $\mathrm{P(WIN)} = 0.94$

Symmetric Equilibrium - Existence Result

Theorem

For any starting budgets, for any technology functions and for any starting point in the campaign, the symmetric strategies form a subgame-perfect equilibrium.

This generalizes Konrad (2016), is analogous to Fu et al. (2015), Klumpp and Polborn (2006) and runs parallel to Klumpp and Konrad (2018).

Proof strategy: two dimensional induction showing symmetric strategies are mutual best responses.

Symmetric Equilibrium - Uniqueness

Theorem

Under the same conditions, the symmetric strategies form the unique equilibrium.

Proof strategy: it follows from Osborne and Rubinstein (1994) that equilibria are 'interchangable' in a fixed-sum game.

Ie if (s_1, s_2) and (s'_1, s'_2) are equilibria then so are (s_1, s'_2) and (s'_1, s_2).

Unique best response thus sufficient.

Discussion

What does the equilibrium imply?

You can have radically assymetric players and still have symmetric equilibrium.

What breaks the equilibrium? Any assymetry in battles.

E.g. point differences, technological differences.

This result is interesting boundary case in those dimensions.

Equilibrium Structure

Recall information set $(i, j, n, b_1, b_2, f, g)$.

Equilibrium (in particular best response) uses only (b_i, n)

So drop the rest and equilibrium remains.

Ie equilibrium from more specific (more info) game 'upwards inducts' to the more general (less info) game.

Equilibrium Structure II

We have the following:

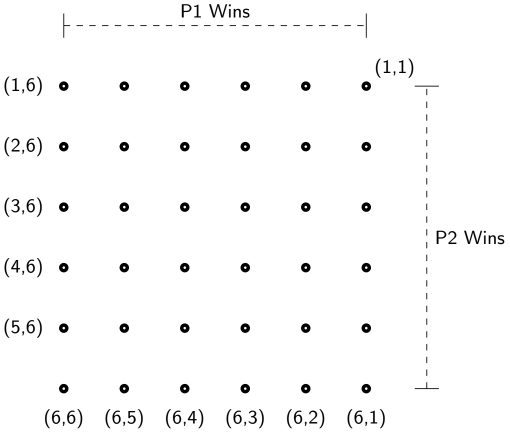
Theorem

The existence of the symmetric equilibrium is:

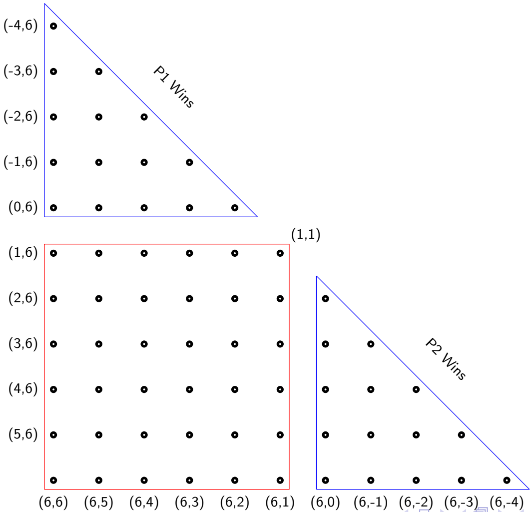
- *Technology and budget independent: i.e. robust to full or partial uncertainty about the technology or budget of a the opponent, or even their own technology;*
- *Temporal-structure independent: i.e. robust to partial or full simultaneity of battles.*

Generalize the game

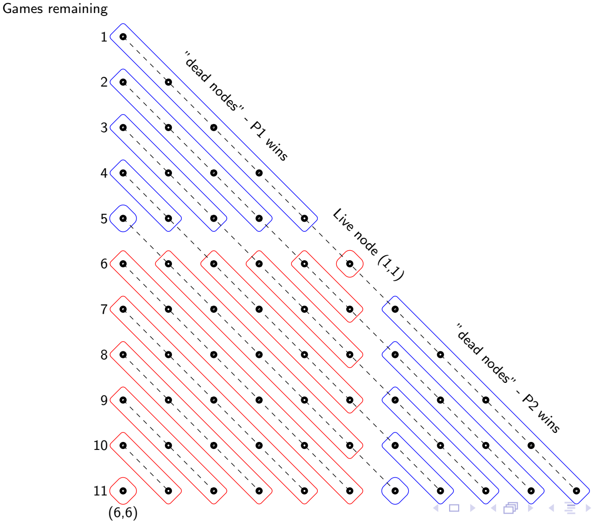
Basic sequential game: State space $= (i, j, n, b_1, b_2, f, g)$



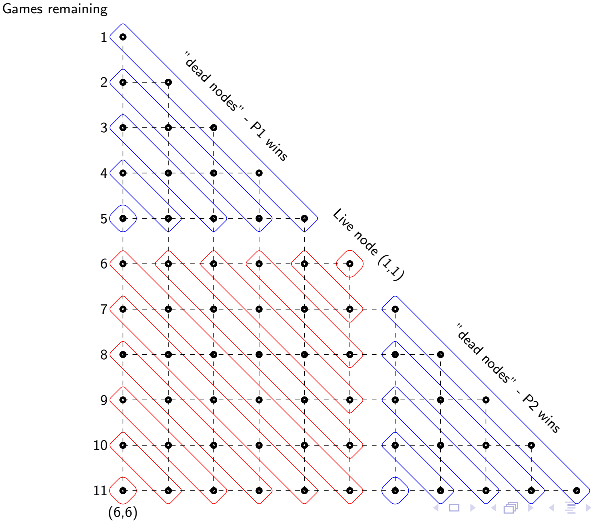
Basic sequential game with all rounds played



Stage Game: state space $= (n, b_1, b_2, f, g)$



Simultaneous Game: state space $= (n, b_i, f, g)$



Upward Induction

Corollary

The symmetric strategies constitute the unique equilibrium of both the stage campaign and the simultaneous campaign.

Proof sketch: suppose opponent is playing symmetrically.

Suppose best response using small state space was not symmetrical.

Then such a response would also have been optimal with larger (but ignorable) state space.

That is, then it would have been a best response in sequential game. Contradiction!

The End

Extensions:

▸ Endogenous Budgets

▸ Asymptotic Structure

Thanks for listening!

Endogenous Budget Choice

Up to now budget choices have been endogenous.

Consider the following game:

1. Players raise money at (convex, increasing) cost functions $c(\cdot)$.
2. Players compete in the first to $2m - 1$ game from above.

Contest length and fundraising

Pure strategy equilibria only exist for small m, for different reasons.

Theorem

If the cost functions are symmetric then the "quasi best response" budget choices b_i^* tend to infinity for each $i = 1, 2$ as $m \to \infty$. Conversely, if the cost functions are asymmetric, then the b_i^* tend to zero for each $i = 1, 2$ as $m \to \infty$.

But in either case, as m gets large, only mixed strategies will exist.

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Asymptotics

We can get a sense of the dynamics as m gets large:

Theorem

The campaign win function for player 1, considered as a sequence of functions in m, converges pointwise to the payoff function for an all pay auction

$$W_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 > b_2, \\ 1/2 & \text{if } b_1 = b_2, \\ 0 & \text{if } b_1 < b_2. \end{cases}$$

Unique mixed equilibrium to the asymptotic game

Theorem

When cost functions are symmetric, each player plays the mixed strategy

$$G(b) = c(b).$$

When cost functions are assymetric, the efficient fundraiser plays $G_1(b) = c_2(b)$ *and in the inefficient fundraiser plays*

$$G_2(b_2) = \underbrace{c_1(b_2)}_{\text{"uniform"}} + \underbrace{c_1(v_1) - c_1(v_2)}_{\text{atom at } 0}.$$

Participation and Fundraising

Corollary

The sum of expected bids, and the probability of participation in the campaign, are higher under symmetric cost functions than asymmetric cost functions.

This is natural, and has important implications for fundraising laws.

Potential empirical work here? But lots of confounding factors.

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Empirical Predictions

Predictions about races between disparate fundraisers

Do strong fundraising incumbents not fundraise and not get challenged ?

Heroic assumptions:

- Fundraising only once
- 'Intrinsic fundraising capacity'

Future directions

Convergence of equilibria instead of just convergence of games.

Strategic vs fundraising effect in two stage game.

Multiple players

Computational methods to check uniqueness and perhaps extend?

Thanks for listening!

Foundations of the CSF

Following Corchón and Dahm (2010), there is an external decider ("voters").

There is a state of the world $\Theta \sim \mathit{Uniform}[0, 1]$

θ high means voters biased right, θ low biased left.

Candidates spend money on advertising. Technology function is effectiveness of campaign.

Foundations of the CSF

If voters have multiplicative utility

$$U_1(x, \theta) = (1 - \theta) f(x), \quad U_2(y, \theta) = \theta g(y)$$

then player 1 is chosen deterministically with probability

$$p_1(x, y) = \frac{f(x)}{f(x) + g(y)}.$$

Back to model

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